

ON THE CARLEMAN ULTRADIFFERENTIABLE VECTORS OF A SCALAR TYPE SPECTRAL OPERATOR

MARAT V. MARKIN

To Academician Yu. M. Berezansky in honor of his 90th jubilee

ABSTRACT. A description of the Carleman classes of vectors, in particular the Gevrey classes, of a scalar type spectral operator in a reflexive complex Banach space is shown to remain true without the reflexivity requirement. A similar nature description of the entire vectors of exponential type, known for a normal operator in a complex Hilbert space, is generalized to the case of a scalar type spectral operator in a complex Banach space.

Never cut what you can untie.
Joseph Joubert

1. INTRODUCTION

The description of the *Carleman classes* of ultradifferentiable vectors, in particular the *Gevrey classes*, of a *normal operator* in a complex Hilbert space in terms of its *spectral measure* established in [10] (see also [12] and [11]) is generalized in [15, Theorem 3.1] to the case of a *scalar type spectral operator* in a complex *reflexive* Banach space.

Here, the reflexivity requirement is shown to be superfluous and a similar nature description of the entire vectors of exponential type, known for a normal operator in a complex Hilbert space (see, e.g., [12]), is generalized to the case of a scalar type spectral operator in a complex Banach space.

2. PRELIMINARIES

For the reader's convenience, we shall outline in this section certain essential preliminaries.

2.1. Scalar type spectral operators. Henceforth, unless specified otherwise, A is supposed to be a *scalar type spectral operator* in a complex Banach space $(X, \|\cdot\|)$ and $E_A(\cdot)$ to be its *spectral measure* (the *resolution of the identity*), the operator's *spectrum* $\sigma(A)$ being the *support* for the latter [1, 4].

In a complex Hilbert space, the scalar type spectral operators are precisely those similar to the *normal* ones [23].

A scalar type spectral operator in complex Banach space has an *operational calculus* analogous to that of a *normal operator* in a complex Hilbert space [1, 3, 4]. To any Borel measurable function $F : \mathbb{C} \rightarrow \mathbb{C}$ (or $F : \sigma(A) \rightarrow \mathbb{C}$, \mathbb{C} is the *complex plane*), there corresponds a scalar type spectral operator

$$F(A) := \int_{\mathbb{C}} F(\lambda) dE_A(\lambda) = \int_{\sigma(A)} F(\lambda) dE_A(\lambda)$$

2010 *Mathematics Subject Classification.* Primary 47B40; Secondary 47B15.

Key words and phrases. Scalar type spectral operator, normal operator, Carleman classes of vectors.

defined as follows:

$$F(A)f := \lim_{n \rightarrow \infty} F_n(A)f, \quad f \in D(F(A)),$$

$$D(F(A)) := \left\{ f \in X \mid \lim_{n \rightarrow \infty} F_n(A)f \text{ exists} \right\}$$

($D(\cdot)$ is the *domain* of an operator), where

$$F_n(\cdot) := F(\cdot) \chi_{\{\lambda \in \sigma(A) \mid |F(\lambda)| \leq n\}}(\cdot), \quad n \in \mathbb{N},$$

($\chi_\delta(\cdot)$ is the *characteristic function* of a set $\delta \subseteq \mathbb{C}$, $\mathbb{N} := \{1, 2, 3, \dots\}$ is the set of *natural numbers*) and

$$F_n(A) := \int_{\sigma(A)} F_n(\lambda) dE_A(\lambda), \quad n \in \mathbb{N},$$

are *bounded scalar type spectral operators* on X defined in the same manner as for a *normal operator* (see, e.g., [3, 21]).

In particular,

$$(2.1) \quad A^n = \int_{\mathbb{C}} \lambda^n dE_A(\lambda) = \int_{\sigma(A)} \lambda^n dE_A(\lambda), \quad n \in \mathbb{Z}_+,$$

($\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ is the set of *nonnegative integers*).

The properties of the *spectral measure* $E_A(\cdot)$ and the *operational calculus*, exhaustively delineated in [1, 4], underly the entire subsequent discourse. Here, we shall outline a few facts of particular importance.

Due to its *strong countable additivity*, the spectral measure $E_A(\cdot)$ is *bounded* [2, 4], i.e., there is such an $M > 0$ that, for any Borel set $\delta \subseteq \mathbb{C}$,

$$(2.2) \quad \|E_A(\delta)\| \leq M.$$

The notation $\|\cdot\|$ has been recycled here to designate the norm in the space $L(X)$ of all bounded linear operators on X . We shall adhere to this rather common economy of symbols in what follows adopting the same notation for the norm in the *dual space* X^* as well.

For any $f \in X$ and $g^* \in X^*$, the *total variation* $v(f, g^*, \cdot)$ of the complex-valued Borel measure $\langle E_A(\cdot)f, g^* \rangle$ ($\langle \cdot, \cdot \rangle$ is the *pairing* between the space X and its dual X^*) is a *finite* positive Borel measure with

$$(2.3) \quad v(f, g^*, \mathbb{C}) = v(f, g^*, \sigma(A)) \leq 4M\|f\|\|g^*\|$$

(see, e.g., [15]). Also (Ibid.), $F : \mathbb{C} \rightarrow \mathbb{C}$ (or $F : \sigma(A) \rightarrow \mathbb{C}$) being an arbitrary Borel measurable function, for any $f \in D(F(A))$, $g^* \in X^*$, and an arbitrary Borel set $\sigma \subseteq \mathbb{C}$,

$$(2.4) \quad \int_{\sigma} |F(\lambda)| dv(f, g^*, \lambda) \leq 4M\|E_A(\sigma)F(A)f\|\|g^*\|.$$

In particular,

$$(2.5) \quad \int_{\mathbb{C}} |F(\lambda)| dv(f, g^*, \lambda) = \int_{\sigma(A)} |F(\lambda)| dv(f, g^*, \lambda) \leq 4M\|F(A)f\|\|g^*\|.$$

The constant $M > 0$ in (2.3)–(2.5) is from (2.2).

The following statement allowing to characterize the domains of the Borel measurable functions of a scalar type spectral operator in terms of positive Borel measures is also fundamental for our discussion.

Proposition 2.1. ([14, Proposition 3.1]). *Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$ and $F : \mathbb{C} \rightarrow \mathbb{C}$ (or $F : \sigma(A) \rightarrow \mathbb{C}$) be Borel measurable function. Then $f \in D(F(A))$ iff*

$$(i) \text{ For any } g^* \in X^*, \int_{\sigma(A)} |F(\lambda)| dv(f, g^*, \lambda) < \infty.$$

$$(ii) \quad \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid |F(\lambda)| > n\}} |F(\lambda)| \, dv(f, g^*, \lambda) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Subsequently, the frequent terms "*spectral measure*" and "*operational calculus*" will be abbreviated to *s.m.* and *o.c.*, respectively.

2.2. The Carleman classes of vectors. Let A be a densely defined closed linear operator in a complex Banach space $(X, \|\cdot\|)$ and $\{m_n\}_{n=0}^\infty$ be a sequence of positive numbers and

$$C^\infty(A) := \bigcap_{n=0}^\infty D(A^n).$$

The subspaces of $C^\infty(A)$

$$C_{\{m_n\}}(A) := \{f \in C^\infty(A) \mid \exists \alpha > 0 \exists c > 0 : \|A^n f\| \leq c \alpha^n m_n, \ n \in \mathbb{Z}_+\},$$

$$C_{(m_n)}(A) := \{f \in C^\infty(A) \mid \forall \alpha > 0 \exists c > 0 : \|A^n f\| \leq c \alpha^n m_n, \ n \in \mathbb{Z}_+\}$$

are called the *Carleman classes* of ultradifferentiable vectors of the operator A corresponding to the sequence $\{m_n\}_{n=0}^\infty$ of Roumieu and Beurling type, respectively.

The inclusions

$$(2.6) \quad C_{(m_n)}(A) \subseteq C_{\{m_n\}}(A) \subseteq C^\infty(A) \subseteq X$$

are obvious.

If two sequences of positive numbers $\{m_n\}_{n=0}^\infty$ and $\{m'_n\}_{n=0}^\infty$ are related as follows:

$$\forall \gamma > 0 \exists c = c(\gamma) > 0 : m'_n \leq c \gamma^n m_n, \quad n \in \mathbb{Z}_+,$$

we also have the inclusion

$$(2.7) \quad C_{\{m'_n\}}(A) \subseteq C_{(m_n)}(A),$$

the sequences being subject to the condition

$$\exists \gamma_1, \gamma_2 > 0, \exists c_1, c_2 > 0 : c_1 \gamma_1^n m_n \leq m'_n \leq c_2 \gamma_2^n m_n, \quad n \in \mathbb{Z}_+,$$

their corresponding Carleman classes coincide

$$(2.8) \quad C_{\{m_n\}}(A) = C_{\{m'_n\}}(A), \quad C_{(m_n)}(A) = C_{(m'_n)}(A).$$

Considering *Stirling's formula* and the latter,

$$\mathcal{E}^{\{\beta\}}(A) := C_{\{[n!]^\beta\}}(A) = C_{\{n^{\beta n}\}}(A),$$

$$\mathcal{E}^{(\beta)}(A) := C_{([n!]^\beta)}(A) = C_{(n^{\beta n})}(A)$$

with $\beta \geq 0$ are the well-known *Gevrey classes* of strongly ultradifferentiable vectors of A of order β of Roumieu and Beurling type, respectively (see, e.g., [10, 11, 12]). In particular, $\mathcal{E}^{\{1\}}(A)$ and $\mathcal{E}^{(1)}(A)$ are the well-known classes of *analytic* and *entire* vectors of A , respectively [6, 19]; $\mathcal{E}^{\{0\}}(A)$ and $\mathcal{E}^{(0)}(A)$ (i.e., the classes $C_{\{1\}}(A)$ and $C_{(1)}(A)$) corresponding to the sequence $m_n \equiv 1$) are the classes of *entire* vectors of *exponential* and *minimal exponential type*, respectively (see, e.g., [22, 12]).

If the sequence of positive numbers $\{m_n\}_{n=0}^\infty$ satisfies the condition

$$(2.9) \quad (\text{WGR}) \quad \forall \alpha > 0 \exists c = c(\alpha) > 0 : c \alpha^n \leq m_n, \quad n \in \mathbb{Z}_+,$$

the scalar function

$$(2.10) \quad T(\lambda) := m_0 \sum_{n=0}^\infty \frac{\lambda^n}{m_n}, \quad \lambda \geq 0, \quad (0^0 := 1)$$

first introduced by S. Mandelbrojt [13], is well-defined (cf. [12]). The function is *continuous*, *strictly increasing*, and $T(0) = 1$.

As is shown in [10] (see also [12] and [11]), the sequence $\{m_n\}_{n=0}^{\infty}$ satisfying the condition **(WGR)**, for a *normal operator* A in a complex Hilbert space X , the equalities

$$(2.11) \quad \begin{aligned} C_{\{m_n\}}(A) &= \bigcup_{t>0} D(T(t|A|)), \\ C_{(m_n)}(A) &= \bigcap_{t>0} D(T(t|A|)) \end{aligned}$$

are true, the normal operators $T(t|A|)$, $t > 0$, defined in the sense of the operational calculus for a normal operator (see, e.g., [3, 21]) and the function $T(\cdot)$ being replaceable with any *nonnegative, continuous, and increasing* on $[0, \infty)$ function $F(\cdot)$ satisfying

$$(2.12) \quad c_1 F(\gamma_1 \lambda) \leq T(\lambda) \leq c_2 F(\gamma_2 \lambda), \quad \lambda \geq R,$$

with some $\gamma_1, \gamma_2, c_1, c_2 > 0$ and $R \geq 0$, in particular, with

$$S(\lambda) := m_0 \sup_{n \geq 0} \frac{\lambda^n}{m_n}, \quad \lambda \geq 0, \quad \text{or} \quad P(\lambda) := m_0 \left[\sum_{n=0}^{\infty} \frac{\lambda^{2n}}{m_n^2} \right]^{1/2}, \quad \lambda \geq 0,$$

(cf. [12]).

In [15, Theorem 3.1], the above is generalized to the case of a *scalar type spectral operator* A in a *reflexive* complex Banach space X . The reflexivity requirement dropped, proved were the inclusions

$$(2.13) \quad \begin{aligned} C_{\{m_n\}}(A) &\supseteq \bigcup_{t>0} D(T(t|A|)), \\ C_{(m_n)}(A) &\supseteq \bigcap_{t>0} D(T(t|A|)) \end{aligned}$$

only, which is a deficiency for statements like [16, Theorem 5.1] and [18, Theorem 3.2].

3. THE CARLEMAN CLASSES OF A SCALAR TYPE SPECTRAL OPERATOR

Theorem 3.1. *Let $\{m_n\}_{n=0}^{\infty}$ be a sequence of positive numbers satisfying the condition **(WGR)** (see (2.9)). Then, for a scalar type spectral operator A in a complex Banach space $(X, \|\cdot\|)$, equalities (2.11) are true, the scalar type spectral operators $T(t|A|)$, $t > 0$, defined in the sense of the operational calculus for a scalar type spectral operator and the function $T(\cdot)$ being replaceable with any nonnegative, continuous, and increasing on $[0, \infty)$ function $F(\cdot)$ satisfying (2.12).*

Proof. We are only to prove the inclusions inverse to (2.13), the rest, including the latter, having been proved in [15, Theorem 3.1].

Consider an arbitrary vector $f \in C_{\{m_n\}}(A)$ ($f \in C_{(m_n)}(A)$). Then necessarily, $f \in C^\infty(A)$ and for a certain $\alpha > 0$ (an arbitrary $\alpha > 0$), there is a $c > 0$ such that

$$(3.14) \quad \|A^n f\| \leq c \alpha^n m_n, \quad n \in \mathbb{Z}_+,$$

(see Preliminaries).

For any $g^* \in X^*$,

$$(3.15) \quad \begin{aligned} \int_{\sigma(A)} T\left(\frac{1}{2\alpha}|\lambda|\right) dv(f, g^*, \lambda) &= \int_{\sigma(A)} \sum_{n=0}^{\infty} \frac{|\lambda|^n}{2^n \alpha^n m_n} dv(f, g^*, \lambda) \\ &\quad \text{by the Monotone Convergence Theorem;} \\ &= \sum_{n=0}^{\infty} \int_{\sigma(A)} \frac{|\lambda|^n}{2^n \alpha^n m_n} dv(f, g^*, \lambda) = \sum_{n=0}^{\infty} \frac{1}{2^n \alpha^n m_n} \int_{\sigma(A)} |\lambda|^n dv(f, g^*, \lambda) \\ &\quad \text{by (2.5) and (2.1);} \end{aligned}$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{2^n \alpha^n m_n} 4M \|A^n f\| \|g^*\|$$

by (3.14);

$$\leq 4Mc \sum_{n=0}^{\infty} \frac{1}{2^n} \|g^*\| = 8Mc \|g^*\| < \infty.$$

For an arbitrary $\varepsilon > 0$, one can fix an $N \in \mathbb{N}$ such that,

$$(3.16) \quad \frac{M^2 c}{2^{N-2}} < \varepsilon/2.$$

Due to the strong continuity of the *s.m.*, for any $n \in \mathbb{N}$,

$$\|E_A(\{\lambda \in \sigma(A) | T(\frac{1}{2\alpha}|\lambda|) > k\}) A^n f\| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Hence, there is a $K \in \mathbb{N}$ such that

$$(3.17) \quad \sum_{n=0}^N \frac{1}{2^n \alpha^n m_n} 4M \|E_A(\{\lambda \in \sigma(A) | T(\frac{1}{2\alpha}|\lambda|) > k\}) A^n f\| < \varepsilon/2$$

whenever $k \geq K$.

Similarly to (3.15), for $k \geq K$, we have

$$\begin{aligned} & \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid T(\frac{1}{2\alpha}|\lambda|) > k\}} T(\frac{1}{2\alpha}|\lambda|) dv(f, g^*, \lambda) \\ &= \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{n=0}^{\infty} \frac{1}{2^n \alpha^n m_n} \int_{\{\lambda \in \sigma(A) \mid T(\frac{1}{2\alpha}|\lambda|) > k\}} |\lambda|^n dv(f, g^*, \lambda) \end{aligned}$$

by (2.4) and (2.1);

$$\begin{aligned} & \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{n=0}^{\infty} \frac{1}{2^n \alpha^n m_n} 4M \|E_A(\{\lambda \in \sigma(A) | T(\frac{1}{2\alpha}|\lambda|) > k\}) A^n f\| \|g^*\| \\ & \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \left[\sum_{n=0}^N \frac{1}{2^n \alpha^n m_n} 4M \|E_A(\{\lambda \in \sigma(A) | T(\frac{1}{2\alpha}|\lambda|) > k\}) A^n f\| \|g^*\| \right. \\ & \quad \left. + \sum_{n=N+1}^{\infty} \frac{1}{2^n \alpha^n m_n} 4M \|E_A(\{\lambda \in \sigma(A) | T(\frac{1}{2\alpha}|\lambda|) > k\})\| \|A^n f\| \|g^*\| \right] \end{aligned}$$

by (2.2) and (3.14);

$$\begin{aligned} & \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \left[\sum_{n=0}^N \frac{1}{2^n \alpha^n m_n} 4M \|E_A(\{\lambda \in \sigma(A) | T(\frac{1}{2\alpha}|\lambda|) > k\}) A^n f\| \|g^*\| \right. \\ & \quad \left. + 4M^2 c \sum_{n=N+1}^{\infty} \frac{1}{2^n} \|g^*\| \right] \end{aligned}$$

$$\leq \sum_{n=0}^N \frac{1}{2^n \alpha^n m_n} 4M \|E_A(\{\lambda \in \sigma(A) | T(\frac{1}{2\alpha}|\lambda|) > k\}) A^n f\| + \frac{M^2 c}{2^{N-2}}$$

by (3.16) and (3.17);

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and we conclude that

$$(3.18) \quad \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\left\{\lambda \in \sigma(A) \mid T\left(\frac{1}{2\alpha}|\lambda|\right) > k\right\}} T\left(\frac{1}{2\alpha}|\lambda|\right) dv(f, g^*, \lambda) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

By Proposition 2.1, (3.15) and (3.18) imply

$$f \in D\left(T\left(\frac{1}{2\alpha}|A|\right)\right).$$

Considering that for $f \in C_{\{m_n\}}(A)$, $\alpha > 0$ is fixed and for $f \in C_{(m_n)}(A)$, $\alpha > 0$ is arbitrary, we infer that

$$f \in \bigcup_{t>0} D(T(t|A|))$$

in the former case and

$$f \in \bigcap_{t>0} D(T(t|A|))$$

in the latter.

Since $f \in C_{\{m_n\}}(A)$ ($f \in C_{(m_n)}(A)$) is arbitrary, we have proved the inclusions

$$C_{\{m_n\}}(A) \subseteq \bigcup_{t>0} D(T(t|A|)),$$

$$C_{(m_n)}(A) \subseteq \bigcap_{t>0} D(T(t|A|)),$$

which along with their inverses (2.13) imply equalities (2.11) to be true. \square

4. THE GEVREY CLASSES

The sequence $m_n := [n!]^\beta$ ($m_n := n^{\beta n}$) with $\beta > 0$ satisfying the condition **(WGR)** and the corresponding function $T(\cdot)$ being replaceable with $F(\lambda) = e^{\lambda^{1/\beta}}$, $\lambda \geq 0$, (see [15] for details, cf. also [12]), in [15, Corollary 4.1] describing the Gevrey classes of vectors of a scalar type spectral operator in a *reflexive* complex Banach space, the reflexivity requirement can be dropped as well and we have the following

Corollary 4.1. *Let $\beta > 0$. Then, for a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$,*

$$\mathcal{E}^{\{\beta\}}(A) = \bigcup_{t>0} D(e^{t|A|^{1/\beta}}),$$

$$\mathcal{E}^{(\beta)}(A) = \bigcap_{t>0} D(e^{t|A|^{1/\beta}}).$$

Corollary 4.1 generalizes the corresponding result of [10] (see also [12, 11]) for a *normal operator* A in a complex Hilbert space and, for $\beta = 1$, gives a description of the *analytic* and *entire* vectors of a scalar type spectral operator A in a complex Banach space.

5. THE ENTIRE VECTORS OF EXPONENTIAL TYPE

Observe that the sequence $m_n \equiv 1$ generating the entire vectors of exponential type does not meet the condition **(WGR)** (see (2.9)) and thus, this case falls outside the realm of Theorem 3.1.

As is known (cf., e.g., [12, 7]), for a *normal operator* A in a complex Hilbert space X ,

$$\mathcal{E}^{\{0\}}(A) = \bigcup_{\alpha>0} E_A(\Delta_\alpha)X$$

and

$$\mathcal{E}^{(0)}(A) = \bigcap_{\alpha>0} E_A(\Delta_\alpha)X = E_A(\{0\})X = \ker A := \{f \in X \mid Af = 0\}$$

($E_A(\cdot)$ is the *spectral measure* of A) with

$$\Delta_\alpha := \{\lambda \in \mathbb{C} \mid |\lambda| \leq \alpha\}, \quad \alpha > 0.$$

We are to generalize the above to the case of a *scalar type spectral operator* A in a complex Banach space X .

Theorem 5.1. *For a scalar type spectral operator A in a complex Banach space $(X, \|\cdot\|)$,*

$$(i) \mathcal{E}^{\{0\}}(A) = \bigcup_{\alpha>0} E_A(\Delta_\alpha)X,$$

$$(ii) \mathcal{E}^{(0)}(A) = \bigcap_{\alpha>0} E_A(\Delta_\alpha)X = E_A(\{0\})X = \ker A := \{f \in X \mid Af = 0\},$$

where $\Delta_\alpha := \{\lambda \in \mathbb{C} \mid |\lambda| \leq \alpha\}$, $\alpha > 0$.

Proof. Let $f \in \bigcup_{\alpha>0} E_A(\Delta_\alpha)X$ ($f \in \bigcap_{\alpha>0} E_A(\Delta_\alpha)X$), i.e.,

$$f = E_A(\{\lambda \in \mathbb{C} \mid |\lambda| \leq \alpha\})f$$

for some (any) $\alpha > 0$. Then, by the properties of the *o.c.*,

$$f \in C^\infty(A).$$

Furthermore, as follows from the *Hahn-Banach Theorem* and the properties of the *o.c.* (in particular, (2.1)),

$$\begin{aligned} \|A^n f\| &= \left\| \int_{\mathbb{C}} \lambda^n dE_A(\lambda) f \right\| = \left\| \int_{\{\lambda \in \mathbb{C} \mid |\lambda| \leq \alpha\}} \lambda^n dE_A(\lambda) f \right\| \\ &= \sup_{g^* \in X^*, \|g^*\|=1} \left| \left\langle \int_{\{\lambda \in \mathbb{C} \mid |\lambda| \leq \alpha\}} \lambda^n dE_A(\lambda) f, g^* \right\rangle \right| \\ &= \sup_{g^* \in X^*, \|g^*\|=1} \left| \int_{\{\lambda \in \mathbb{C} \mid |\lambda| \leq \alpha\}} \lambda^n d\langle E_A(\lambda) f, g^* \rangle \right| \\ &\leq \sup_{g^* \in X^*, \|g^*\|=1} \int_{\{\lambda \in \mathbb{C} \mid |\lambda| \leq \alpha\}} |\lambda|^n dv(f, g^*, \lambda) \\ &\leq \sup_{g^* \in X^*, \|g^*\|=1} \alpha^n v(f, g^*, \{\lambda \in \mathbb{C} \mid |\lambda| \leq \alpha\}) \\ &\leq \sup_{g^* \in X^*, \|g^*\|=1} 4M \|f\| \|g^*\| \alpha^n \leq 4M [\|f\| + 1] \alpha^n, \quad \alpha > 0, \end{aligned}$$

by (2.3);

which implies that $f \in \mathcal{E}^{\{0\}}(A)$ ($f \in \mathcal{E}^{(0)}(A)$).

Conversely, for an arbitrary $f \in \mathcal{E}^{\{0\}}(A)$ ($f \in \mathcal{E}^{(0)}(A)$),

$$\|A^n f\| \leq c \alpha^n, \quad n \in \mathbb{Z}_+,$$

with some (any) $\alpha > 0$ and some $c > 0$.

Then, for any $\gamma > \alpha$ and $g^* \in X^*$, we have

$$\begin{aligned} \gamma^n v(f, g^*, \{\lambda \in \mathbb{C} \mid |\lambda| \geq \gamma\}) &\leq \int_{\{\lambda \in \mathbb{C} \mid |\lambda| \geq \gamma\}} |\lambda|^n dv(f, g^*, \lambda) \\ &\leq \int_{\mathbb{C}} |\lambda|^n dv(f, g^*, \lambda) \\ &\leq 4M \|A^n f\| \|g^*\| \leq 4Mc \|g^*\| \alpha^n, \quad n \in \mathbb{Z}_+. \end{aligned}$$

by (2.5);

Therefore,

$$v(f, g^*, \{\lambda \in \mathbb{C} \mid |\lambda| \geq \gamma\}) \leq 4Mc \|g^*\| \left(\frac{\alpha}{\gamma}\right)^n, \quad n \in \mathbb{Z}_+.$$

Considering that $\alpha/\gamma < 1$ and passing to the limit as $n \rightarrow \infty$, we conclude that

$$v(f, g^*, \{\lambda \in \mathbb{C} \mid |\lambda| \geq \gamma\}) = 0, \quad g^* \in X^*,$$

and the more so

$$\langle E_A(\{\lambda \in \mathbb{C} \mid |\lambda| \geq \gamma\})f, g^* \rangle = 0, \quad g^* \in X^*.$$

Whence, as follows from the *Hahn-Banach Theorem*,

$$E_A(\{\lambda \in \mathbb{C} \mid |\lambda| \geq \gamma\})f = 0,$$

which, considering that $\gamma > \alpha$ is arbitrary, by the *strong continuity* of the *s.m.*, implies that

$$E_A(\{\lambda \in \mathbb{C} \mid |\lambda| > \alpha\})f = 0.$$

Hence, by the *additivity* of the *s.m.*,

$$f = E_A(\{\lambda \in \mathbb{C} \mid |\lambda| \leq \alpha\})f + E_A(\{\lambda \in \mathbb{C} \mid |\lambda| > \alpha\})f = E_A(\{\lambda \in \mathbb{C} \mid |\lambda| \leq \alpha\})f,$$

which implies that $f \in \bigcup_{\alpha>0} E_A(\Delta_\alpha)X$ ($f \in \bigcap_{\alpha>0} E_A(\Delta_\alpha)X$). \square

An immediate implication of Theorem 5.1 is the following generalization of the well-known result on the denseness of exponential type vectors of a normal operator in a complex Hilbert space (see, e.g., [12]), which readily follows by the *strong continuity* of the *s.m.* and joins a number of similar results of interest for approximation and qualitative theories (see [22, 12, 7, 8, 9]).

Corollary 5.1. *For a scalar-type spectral operator A in a complex Banach space $(X, \|\cdot\|)$,*

$$\overline{\mathcal{E}^{\{0\}}(A)} = X$$

($\overline{\cdot}$ is the closure of a set in the strong topology of X).

Hence, for any positive sequence $\{m_n\}_{n=0}^\infty$ satisfying the condition **(WGR)**, in particular, for $m_n = [n!]^\beta$ with $\beta > 0$, due to inclusion (2.7) with $m'_n \equiv 1$,

$$\mathcal{E}^{\{0\}}(A) \subseteq C_{(m_n)}(A),$$

which implies

$$\overline{C_{(m_n)}(A)} = X.$$

6. FINAL REMARKS

Observe that, for a normal operator in a complex Hilbert, equalities (2.11) have not only the set-theoretic but also a topological meaning [10] (see also [11, 12]). By analogy, this also appears to be true for a scalar type spectral operator in a complex Banach space, although the idea was entertained by the author neither in [15] nor here.

For a normal operator in a complex Hilbert space, Theorems 5.1 and 3.1 can be considered as generalizations of *Paley-Wiener Theorems* relating the smoothness of a square-integrable on the real axis \mathbb{R} function $f(\cdot)$ to the decay of its *Fourier transform* $\hat{f}(\cdot)$ as $x \rightarrow \pm\infty$ [20], which precisely corresponds to the case of the *self-adjoint* differential operator $A = i\frac{d}{dx}$ (i is the *imaginary unit*) in the complex Hilbert space $L_2(\mathbb{R})$ [12]. Observe that, in $L_p(\mathbb{R})$ with $1 \leq p < \infty$, $p \neq 2$, the same operator fails to be spectral [5] (the domain of $A = i\frac{d}{dx}$ in $X = L_p(\mathbb{R})$, $1 \leq p < \infty$, is understood to be the subspace $W_p^1(\mathbb{R}) := \{f \in L_p(\mathbb{R}) \mid f(\cdot) \text{ is absolutely continuous on } \mathbb{R} \text{ and } f' \in L_p(\mathbb{R})\}$).

Theorem 3.1 entirely substantiates the proof of the "*only if*" part of [16, Theorem 5.1], where inclusions (2.13) turn out to be insufficient, and of [18, Theorem 3.2]. It appears to be fundamental for qualitative results of this nature (cf. [17]).

Acknowledgments. The author's utmost appreciation is to Drs. Miroslav L. and Valentina I. Gorbachuk, whose work and life have become a perpetual source of inspiration, ideas, and goodness for him.

REFERENCES

1. N. Dunford, *A survey of the theory of spectral operators*, Bull. Amer. Math. Soc. **64** (1958), 217–274.
2. N. Dunford and J. T. Schwartz with the assistance of W. G. Bade and R. G. Bartle, *Linear Operators*. Part I: *General Theory*, Interscience Publishers, New York, 1958.
3. ———, *Linear Operators*. Part II: *Spectral Theory. Self Adjoint Operators in Hilbert Space*, Interscience Publishers, New York, 1963.
4. ———, *Linear Operators*. Part III: *Spectral Operators*, Interscience Publishers, New York, 1971.
5. R. Farwig and E. Marschall, *On the type of spectral operators and the nonspectrality of several differential operators on L^p* , Integral Equations Operator Theory **4** (1981), no. 2, 206–214.
6. R. Goodman, *Analytic and entire vectors for representations of Lie groups*, Trans. Amer. Math. Soc. **143** (1969), 55–76.
7. M. L. Gorbachuk and V. I. Gorbachuk, *On the approximation of smooth vectors of a closed operator by entire vectors of exponential type*, Ukrain. Mat. Zh. **47** (1995), no. 5, 616–628. (Ukrainian); English transl. Ukrainian Math. J. **47** (1995), no. 5, 713–726.
8. ———, *On the well-posed solvability in some classes of entire functions of the Cauchy problem for differential equations in a Banach space*, Methods Funct. Anal. Topology **11** (2005), no. 2, 113–125.
9. ———, *On completeness of the set of root vectors for unbounded operators*, Methods Funct. Anal. Topology **12** (2006), no. 4, 353–362.
10. V. I. Gorbachuk, *Spaces of infinitely differentiable vectors of a nonnegative self-adjoint operator*, Ukrain. Mat. Zh. **35** (1983), no. 5, 617–621. (Russian); English transl. Ukrainian Math. J. **35** (1983), no. 5, 531–534.
11. V. I. Gorbachuk and M. L. Gorbachuk, *Boundary Value Problems for Operator Differential Equations*, Kluwer Academic Publishers, Dordrecht—Boston—London, 1991. (Russian edition: Naukova Dumka, Kiev, 1984)
12. V. I. Gorbachuk and A. V. Knyazyuk, *Boundary values of solutions of operator-differential equations*, Russian Math. Surveys **44** (1989), no. 3, 67–111.
13. S. Mandelbrojt, *Séries de Fourier et Classes Quasi-Analytiques de Fonctions*, Gauthier-Villars, Paris, 1935.
14. M. V. Markin, *On an abstract evolution equation with a spectral operator of scalar type*, Int. J. Math. Math. Sci. **32** (2002), no. 9, 555–563.
15. ———, *On the Carleman classes of vectors of a scalar type spectral operator*, Ibid. **2004** (2004), no. 60, 3219–3235.
16. ———, *On scalar type spectral operators and Carleman ultradifferentiable C_0 -semigroups*, Ukrain. Mat. Zh. **60** (2008), no. 9, 1215–1233; English transl. Ukrainian Math. J. **60** (2008), no. 9, 1418–1436.
17. ———, *On the Carleman ultradifferentiability of weak solutions of an abstract evolution equation*, Modern Analysis and Applications, Oper. Theory Adv. Appl., vol. 191, pp. 407–443, Birkhäuser Verlag, Basel, 2009.
18. ———, *On the generation of Beurling type Carleman ultradifferentiable C_0 -semigroups by scalar type spectral operators*, Methods Funct. Anal. Topology (to appear).
19. E. Nelson, *Analytic vectors*, Ann. of Math. (2) **70** (1959), no. 3, 572–615.
20. R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, Amer. Math. Soc. Coll. Publ., vol. 19, Amer. Math. Soc., New York, 1934.
21. A. I. Plesner, *Spectral Theory of Linear Operators*, Nauka, Moscow, 1965. (Russian)
22. Ya. V. Radyno, *The space of vectors of exponential type*, Dokl. Akad. Nauk BSSR **27** (1983), no. 9, 791–793. (Russian)
23. J. Wermer, *Commuting spectral measures on Hilbert space*, Pacific J. Math. **4** (1954), no. 3, 355–361.

DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY, FRESNO 5245 N. BACKER AVENUE,
M/S PB 108 FRESNO, CA 93740-8001
E-mail address: mmarkin@csufresno.edu

Received 04/06/2015